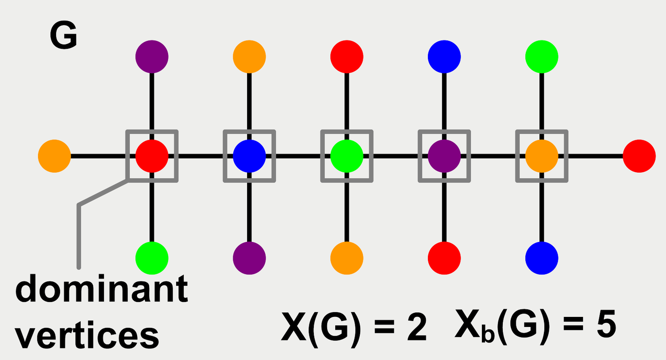
On the *b*-coloring of P_4-tidy graphs

Clara Inés Betancur Velasquez, Flavia Bonomo, Ivo Koch
Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires

DEPARTAMENTO DE COMPUTACION
Facultad de Cs. Exactas y Naturales
UNIVERSIDAD DE BUENOS AIRES

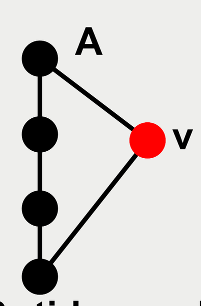
b-chromatic number

- A *coloring* of a graph G is an assignment of colors (represented by natural numbers) to the vertices of G such that any two adjacent vertices are assigned different colors.
- The smallest number t such that G admits a coloring with t colors is called the *chromatic number* of G, and is denoted by $\chi(G)$.
- A ***b**-coloring* of a graph is a coloring C that verifies the following property for every color c_i in C: There is at least one vertex v_i colored with c_i, such that all the other colors in C are used in v_is neighbours. v_i is called a *dominant* vertex.
- The ***b**-chromatic number* of a graph G, noted $\chi_b(G)$, is the maximum number t such that a b-coloring with t colors exists for G.



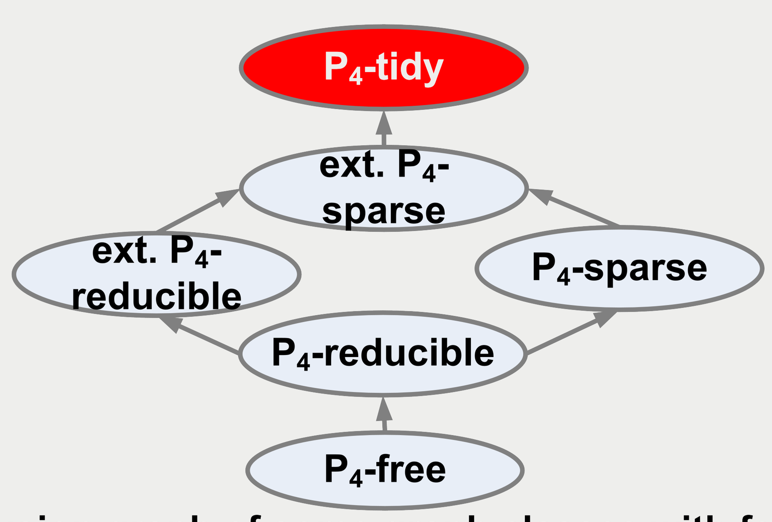
P_4-tidy graphs

- Let A be a P_4 (an induced path of four vertices) in G. A vertex v in G is a *partner* of A if $G[A \cup \{v\}]$ contains at least two P_4's.
- G is called *P_4-tidy* if every P_4 in G has at most one partner.



A P_4-tidy graph

- *Cographs* are the P_4-free graphs, that is, graphs with no P_4s. P_4-tidy graphs are a superclass of cographs, and they also generalize several graph classes with "few"P_4s studied in the literature.

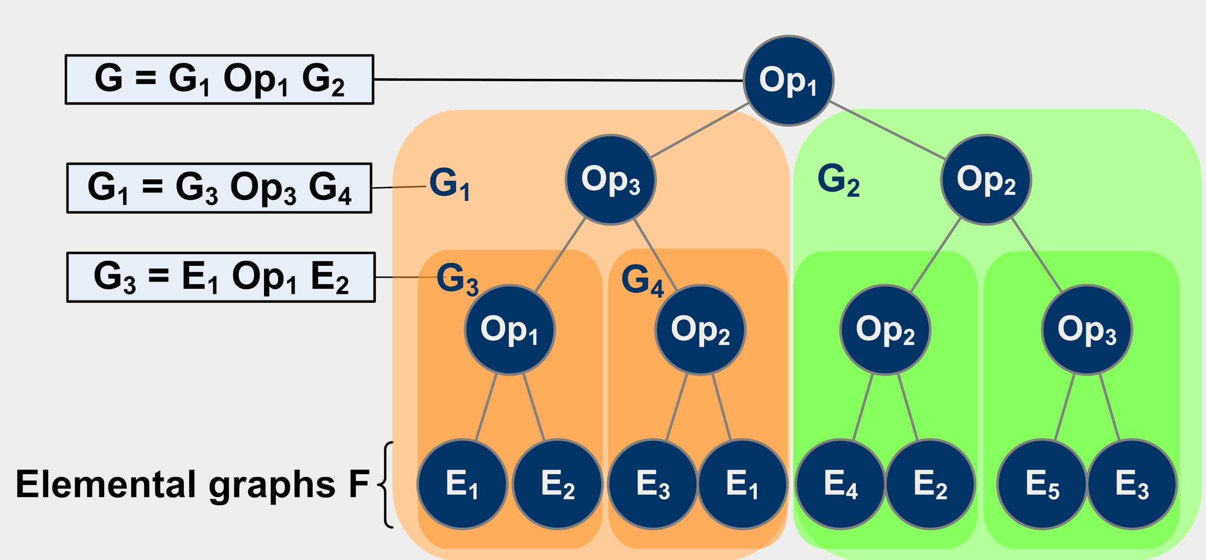


Inclusion graph of some graph classes with few P_4s

An algorithm for the *b*-coloring of P_4-tidy graphs

Theorem. The b-chromatic number of a P_4-tidy graph can be computed by a dynamic programming algorithm in $O(n^3)$.
Main ideas of the algorithm:

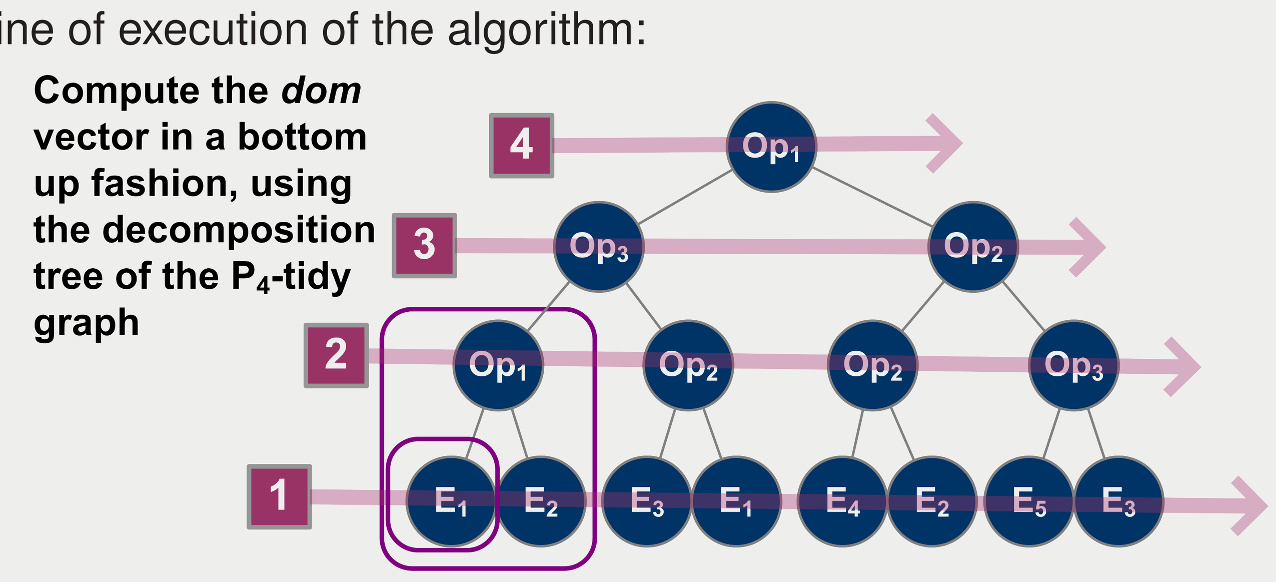
- **Theorem.** Every P_4-tidy graph can be constructed by repeated application of three binary operations (we'll note Op_1, Op_2, Op_3), starting with a family F of elemental graphs.



- Let G be a P_4-tidy graph and dom_G an array of $|V(G)|$ positions, where $dom_G[t]$ is the maximum number of distinct colors containing dominant vertices in any coloring of G with t colors. Note that $\chi_b(G)$ is the greatest t that verifies $dom_G[t] = t$.

  Theorem. There is an effective procedure for:
  1. Computing dom_G for every possible elemental graph in F.
  2. Computing dom_G for $G = G_1 * G_2$, where $* \in \{Op_1, Op_2, Op_3\}$ and G_1, G_2 are two smaller P_4-tidy graphs for which we know dom_{G_1} and dom_{G_2}.

- Outline of execution of the algorithm:

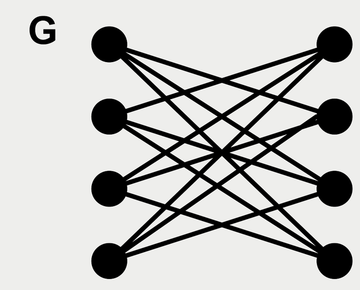


b-continuity of P_4-tidy graphs

The behaviour of the b-chromatic number differs notoriously from the classical chromatic number. The values of k for which a graph admits a b-coloring do not necessarily form an interval in the set of integers. It makes therefore sense to study the graph families in which these values of k form indeed an interval.

A graph is *b-continuous* if it admits a b-coloring with t colors, for $t = \chi(G), \ldots, \chi_b(G)$.

A non b-continuous graph



G admits a b-coloring with 2 colors
G does not admit a b-coloring with 3 colors
G admits a b-coloring with 4 colors

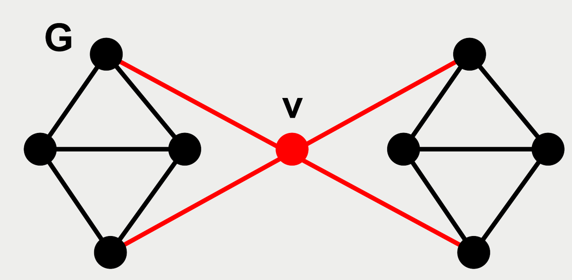
Theorem. P_4-tidy graphs are b-continuous.

b-monotonicity of P_4-tidy graphs

Another atypical property of the b-chromatic number is that it can increase when taking induced subgraphs.

A graph G is b-monotonous if $\chi_b(H_1) \geq \chi_b(H_2)$ for every induced subgraph H_1 of G and every induced subgraph H_2 of H_1.

A non b-monotonous graph

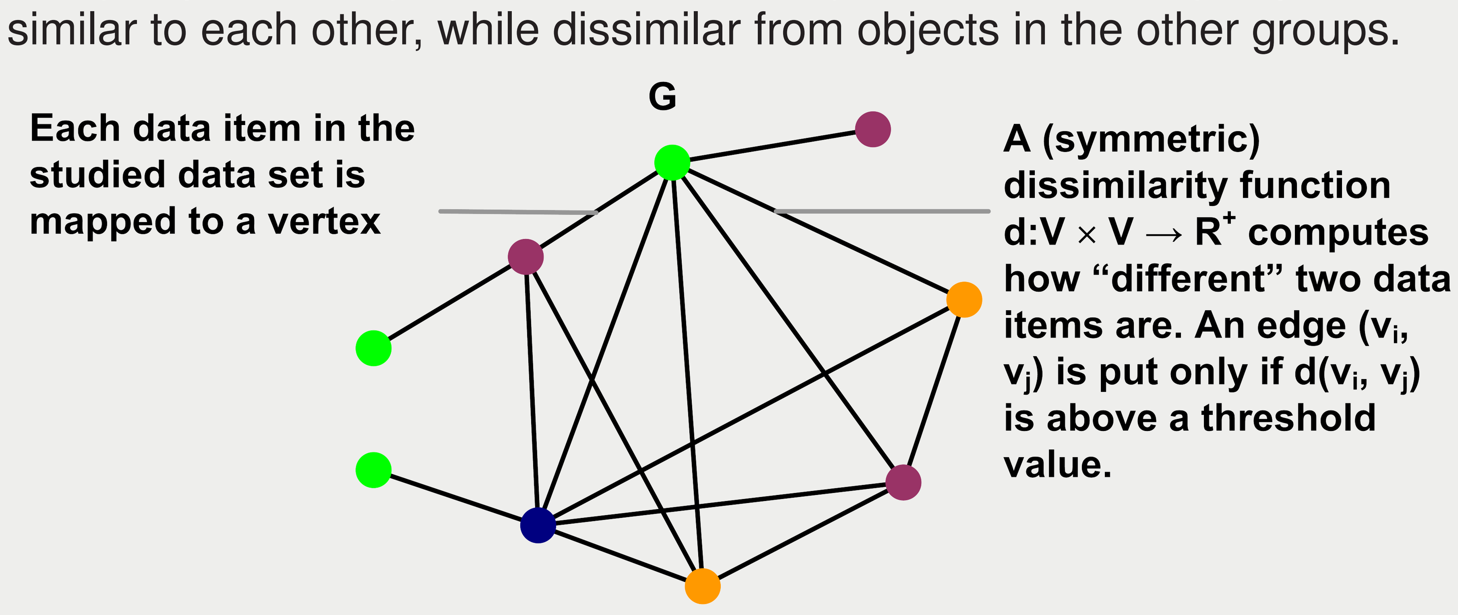


$X_b(G) = 3$, but the subgraph H obtained by deleting v verifies $X_b(H) = 4$

Theorem. P_4-tidy graphs are b-monotonous.

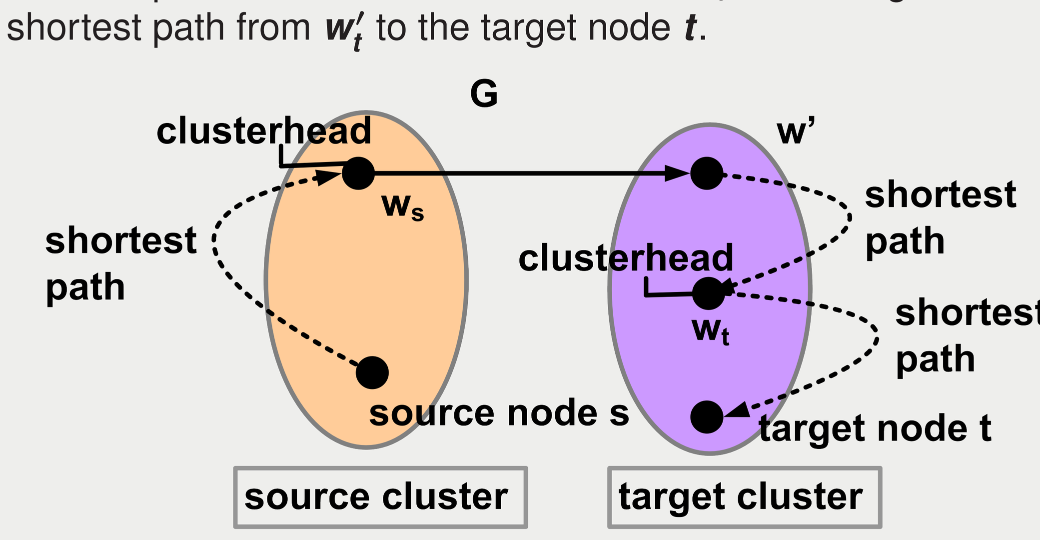
Recent applications of *b*-coloring

- **Clustering in Data Mining applications.** Informally, the problem of Clustering consists of the division of data items (objects, instances, etc.) into groups or categories, such that all objects in the same group are similar to each other, while dissimilar from objects in the other groups.



  A b-coloring of the graph builds clusters of the data (the vertices with the same color) with guaranteed inter cluster differences due to the presence of dominant vertices in each cluster.

- ***b*-coloring based routing method.** b-coloring may also be used for routing information between nodes of a weighted graph. Let s be the source node and t the target node. The idea is to decompose the graph in clusters (again, vertices with the same color) using a distributed b-coloring algorithm, and to use the dominant vertex (called 'clusterhead') of the same color of s to join the cluster of t. The steps for determining a path from s to t are:
  1. Find the shortest path from s to the clusterhead w_s.
  2. Join the neighbor w' of w_s in the cluster containing node t (property of the clusterhead w_s).
  3. Find the shortest path from w' to the clusterhead w_t of the target node t.
  4. Find the shortest path from w'_t to the target node t.



Acknowledgements

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